The Derivative

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1. Introduction

The derivative operator is one of the most important concepts in calculus, and one of the fundamental building blocks for all advanced studies regarding STEM. Therefore, it is good to know the basics behind the subject. I will be introducing them from the perspective of infinitesimal number, or the hyperreal number system as it is more formally known.

Imagine that you are a physicist studying rate of change. You know that you can measure the position of a certain object through time, and that therefore:

$$v_{average} = \frac{d_{final} - d_{initial}}{t_{final} - t_{initial}}$$

If you've taken a grade 11 physics class, you should recall that this is because measuring the slope of a line gives you the amount change of d (displacement) in units of t (time), or as the math puts it, the ratio between the change in d (displacement) over t (time) is equal to the average velocity.

As a concrete example, imagine I am walking across a 100m road, with only two directions: forwards and backwards. If I start 20m into the road, and after 10 seconds of walking I end up 30m into the road, my final displacement would be 30m, my initial displacement would be 20m, and the amount of time that elapsed was 10 seconds (my initial amount of time would be 0s into the experiment, and my final amount of time would be 10 seconds). Therefore:

$$v_{average} = \frac{30 - 20}{10 - 0} = 1m/s$$

The above result concludes that on average between the 10 seconds we were walking, we traveled 1m/s. Note that this does not mean that we traveled 1m/s all the time; it could be the case that we traveled 0.5m/s some of the way and 2m/s some other part of the way and it all averaged out to be 1m/s. From here we can ask a very important question: is there a way we can find the velocity such that instead of giving an average value, we can give an exact value for a specific point in time? Or, another way to phrase the question: how to we get the velocity at an instantaneous moment in time instead of over a period of time?

2. The Derivative

In fact, there is a way. Instead of taking the average over some patch of time, we take the average over an *infinitely small* amount of time. In this way, we can calculate the near-instant speed of an object with infinitely small inaccuracy.

Let's take a simple concrete example of what I am describing. Given a simple function $d = t^2$ where$t$ is time and d is displacement, we want to get the instantaneous velocity at a general point (time and position respectively) $(t_{initial}, d_{initial})$ with their inifinitely small changes in distance and time beingΔt and Δd respectively:

$$d_{final} = d_{initial} + \Delta d$$

$$t_{final} = t_{initial} + \Delta t$$

$$v_{average} = \frac{d_{final} - d_{initial}}{t_{final} - t_{initial}}$$

$$v_{average} = \frac{d_{initial} + \Delta d - d_{initial}}{t_{inital} + \Delta t - t_{initial}}$$

$$v_{average} = \frac{\Delta d}{\Delta t}$$

in words, we step through an infinitely small amount of time (Δt) and then we can see what the resulting infinitely small step in displacement is (Δd).

That might make sense, but how do we actually calculate what $\frac{\Delta d}{\Delta t}$ is? We have too many variables, we can't solve for both displacement and time unless we have more information. Well, remember, we have a function that relates the amount of time that passes to the amount of displacement that happens. We can use that in order to get more information about how displacement and time are related so that we can express everything in terms of one variable, time. This next section is focused on replacing all the displacement variables with time variables, kind of like substitution with systems of linear equations.

3. Calculations

$$d = t^2$$

$$d_{initial} = t_{initial}^2$$

$$d_{final} = t_{final}^2$$

and all of this must be true by definition. Given this:

$$d_{final} = d_{initial} + \Delta d$$

so we know that:

$$d_{initial} + \Delta d = t_{final}^2$$

but we also know that $t_{final} = t_{initial} + \Delta t$, so:

$$d_{initial} + \Delta d = (t_{initial} + \Delta t)^2$$

$$\Delta d = (t_{initial} + \Delta t)^2 - d_{initial}$$

but remember:

$$v_{average} = \frac{\Delta d}{\Delta t}$$

so:

$$v_{average} = \frac{(t_{initial} + \Delta t)^2 - d_{initial}}{\Delta t}$$

But:

$$d = t^2$$

So:

$$d_{initial} = t_{initial}^2$$

And, therefore:

$$v_{average} = \frac{(t_{initial} + \Delta t)^2 - t_{initial}^2}{\Delta t}$$

So now we can finally do some math in order to figure out what velocity is, where the rest should just be easy grade 11 mathematics. We can expand the binomial here:

$$v_{average} = \frac{t_{initial}^2 + 2t_{initial}\Delta t + \Delta t^2 - t_{initial}^2}{\Delta t}$$

$$v_{average} = \frac{2t_{initial}\Delta t + \Delta t^2}{\Delta t}$$

$$v_{average} = 2t_{initial} + \Delta t$$

What does this equation mean? It means that this average velocity over an infinitely small period in time is going to grow at a rate of $2t + \Delta t$. In other words, for any point in time, the velocity at that point in time will always be $2t + \Delta t$. First of all, since Δt is infinitely small, we tend to ignore that term and just say the velocity at any given moment in time is going to be $2t$. Second of all, instead of being an average velocity, this starts becoming more like an instantaneous velocity. So we say:

$$v = 2t$$

because we used a general point fot $t_{initial}$, we've shown that it works for every point, so we can represent the resulting instantaneous velocity as a function. Look at what we've just done: we've taken a function of position in terms of time ($d = t^2$) and turned it into a function of *velocity* in terms of time ($v = 2t$). This is what people mean when they say taking a derivative. Since velocity is the measurement of the change of displacement over time, what people mean when they say to take a derivative of a function is to take the rate of change of that function, or how much that function changes over time (or another variable).

4. General Form

In fact, instead of using a specific function $d = t^2$, we can find the general form of a derivative for any function. Suppose we know that $d = f(t)$ where $f(t)$ is our general function relating displacement to time. We can do the same process as before except more abstractly:

$$d_{initial} = f(t_{initial})$$

$$d_{final} = f(t_{final})$$

$$\Delta = f(t_{initial} + \Delta t) - d_{initial}$$

$$\Delta = f(t_{initial} + \Delta t) - f(t_{initial})$$

$$f'(t) = \frac{f(t_{initial} + \Delta t) - f(t_{initial})}{\Delta t}$$

Where $f'(t)$ is the derivative of $f(t)$. Of course, I skipped a lot of steps, b ut all the steps that I skipped we already went over in another form above. I leave the specific details as an excercise.

Of course, we don't just use derivatives for physics, so time sometimes isn't the thing that is changing in our equations. The true definition of a derivative is:

$$f'(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

where x is a general variable we are measuring the change of. Also, note that the *initial* subscript is removed because there is no final or initial x anymore, there's an instantaneous change.

5. Conclusion

Think about what we've done. We started with a way to get the average velocity, we came up with a question regarding instantaneous velocity, then we used an approach using infinitely small numbers in order to solve the problem. Every step of the way started with a less abstract problem, which we then generalized in order to solve more general problems. This is the fundamental skill of getting better at anything, not just mathematics. With that, I think this is a good end to this article.